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On the Trace and the Relativistic Scalar Products Involving Dirac Matrices γ_i and the 4 × 4 Unit Matrix

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Abstract

In this paper we have dealt with the relativistic scalar products like $\gamma_i u \gamma_i$, Tr $(\gamma_i u)$ Tr $(\gamma_i v)$ and Tr $(\gamma_i \gamma_j u)$ Tr $(\gamma_i \gamma_j v)$. Here any string like u and v involve Dirac matrices in the manner $u = \Pi_{r=1}^{n} a_r$ where the element $a_r = a_{r_i}\gamma_i + ia_{r_5}$ i.e., a_r in general involves a term ia_{r_5} which is multiplied by a 4 x 4 unit matrix. We have further evaluated Tr $(\gamma_b u)$ Tr $(\gamma_b v)$ where the suffix 'b', unlike the dummy suffixes i and j, does not imply any summation and can assume any specific value from 1 to 5. Some reduction formulae for the evaluation of Tr $(\gamma_5 u)$ and Tr(u) have been obtained in this paper.

1. *Introduction*

Several authors have worked on the problem of the determination of the relativistic scalar products of Dirac matrixes. This is required in dealing with the problems of quantum electrodynamics and some weak interaction processes with the help of relativistic perturbation theory. We may mention that the results of Chisholm apply when we deal with a string (involving Dirac matrices) of the type $u = u_n = a_1, a_2 \ldots a_n$ where the element $a_r = \Sigma_{i=1}^{4} a_{r_i}\gamma_i$. But we also need to consider the same problem when in general the element $a_r = a_{r_i}\gamma_i + ia_{r_5}$ i.e., a_r involves in general a term ia_{r_5} multiplied by a 4 x 4 unit matrix. As an illustration we may point out that in the trace calculation involved in the perturbation theory we come across terms like $p_i\gamma_i + im$ where m and p_i are the mass and ith component of momenta respectively of a particle In this connection we may mention that in a previous paper (1973) we have dealt with an analogous problem where the element a_r of the string u involves 2 x 2 Pauli matrices and a 2 x 2 unit matrix in the manner $a_r = a_{r_i}\sigma_i + ia_{r_4}$. In order to deal with this above-mentioned problem we have introduced the matrices Γ_μ where $\Gamma_i = i\gamma_i\gamma_5$ and $\Gamma_5 = \gamma_5$. It may be noted that the five Γ_μ matrices, like γ_i and γ_5 matrices, satisfy the well-known anti-commutation relation. We then define a new string $U = U_n = A_1A_2 \ldots A_n = \Pi_{r=1}^n A_r$ where the element $A_r = A_{r_\mu}\Gamma_\mu$. It has been shown here that the problem of the trace calculation involving γ_i, γ_5 matrices and string like u can be related to one involving Γ_μ matrices and string like U. A relation between a_{r_5} and A_{r_5} which occur in the strings u and U respectively has been set up. In this paper we have evaluated the relativistic scalar products like $\gamma_i u \gamma_i$ where u $(= u_n)$ may be an even or odd string, i.e. n may be even or odd. We have also dealt with other types of relativistic scalar products like $(\gamma_i u)(\gamma_i v)$ and $(\gamma_i\gamma_j u)(\gamma_i\gamma_j v)$. Here we have used the abbreviated notation of the trace bracket ( ) like Tr $(u) = (u)$. We have further evaluated the expression $(\gamma_b u)(\gamma_b v)$ where the repeated suffix 'b', unlike the dummy suffixes i and j, can assume any specific value from 1 to 5 instead of implying a summation. In particular the product of the two traces $(\gamma_5 u)(\gamma_5 v)$ is expressed in a form which does not involve any γ_5 matrix. Some reduction formulae for the determination of the traces like (u) and $(\gamma_5 u)$ have been derived in this paper.

2. *Calculation*

The Γ_μ matrices defined by

$$\Gamma_i = i\gamma_i\gamma_5, \qquad \Gamma_5 = \gamma_5 \tag{2.1}$$

satisfy the following anti-commutation relation

$$\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu = 2\delta_{\mu\nu} \tag{2.2}$$

In (2.1) and (2.2) and in the following Greek suffixes μ, ν, etc. stand for any number from 1 to 5, while Latin suffixes i, j, etc. for any value from 1 to 4.

The string U of Γ_μ matrices is written as

$$U = U_n = A_1A_2 \ldots A_n = \prod_{r=1}^{n} A_r \tag{2.3}$$

where the element A_r is given by

$$A_r = A_{r_i}\Gamma_i + A_{r_5}\Gamma_5 = A_{r_\mu}\Gamma_\mu \tag{2.4}$$

Similarly the string u involving γ_i and 4×4 unit matrices is given by

$$u = u_n = a_1a_2 \ldots a_n = \prod_{r=1}^{n} a_r \tag{2.5}$$

where the element a_r is given by

$$a_r = a_{r_i}\gamma_i + ia_{r_5} \tag{2.6}$$

We also define

$$\bar{U} = (-1)^n \Gamma_5 U_n \Gamma_5 = \prod_{r=1}^{n} \bar{A}_r \tag{2.7}$$

and

$$\bar{u} = (-1)^n \gamma_5 u_n \gamma_5 = \prod_{r=1}^{n} \bar{a}_r \tag{2.8}$$

where

$$\bar{A}_{r_i} = A_{r_i}, \qquad \bar{A}_{r_5} = -A_{r_5} \tag{2.9}$$

and

$$\bar{a}_{r_i} = a_{r_i}, \qquad \bar{a}_{r_5} = -a_{r_5} \tag{2.10}$$

so that the notation of 'bar' implies that the sign of the fifth components of the elements is to be reversed. Throughout this paper we use the abbreviated notation of trace bracket like

$$\mathrm{Tr}\,(U) = (U) \tag{2.11}$$

Since we can write

$$a_r = -i\,[a_{r_i}\Gamma_i - a_{r_5}\Gamma_5]\Gamma_5 \tag{2.12}$$

we are able to set up the following relations between the strings defined by (2.3) and (2.5)

$$(U_n) = (u_n) \tag{2.13}$$

when n is even and

$$(U_n) = i(\gamma_5 u_n) \tag{2.14}$$

when n is odd.

In (2.13) and (2.14) the elements of U_n and u_n are related in the following manner

$$A_{r_i} = a_{r_i}, \qquad A_{r_5} = (-1)^r a_{r_5} \tag{2.15}$$

Now with the aid of relation (2.1) one can express the sixteen independent elements of Dirac algebra by the elements $\frac{1}{2}[\Gamma_\mu\Gamma_\nu - \Gamma_{\nu\gamma}\Gamma_\mu]$ where $(\mu > \nu)$, Γ_μ and 4 x 4 unit matrix. Here the suffixes μ and ν are assumed to take any value from 1 to 5. Using this fact and the anti-commutation relation (2.2) one can easily verify the following relation involving the string U_n

$$\begin{aligned}\tfrac{1}{8}([\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu]\,U_n)[\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu] + (\Gamma_\mu U_n)\Gamma_\mu \\ = \tfrac{1}{8}([\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu]\,\Gamma_b U_n)[\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu]\,\Gamma_b + (\Gamma_b U_n)\Gamma_b \\ + \tfrac{1}{4}([\Gamma_\mu\Gamma_b - \Gamma_b\Gamma_\mu]\,U_n)[\Gamma_\mu\Gamma_b - \Gamma_b\Gamma_\mu]\end{aligned} \tag{2.16}$$

where the particular suffix 'b' in Γ_b, unlike the dummy suffixes μ and ν, though repeated, does not imply any summation. The suffix 'b' may assume any specific value from 1 to 5.

Using relation (2.2) we obtain from (2.16)

$$Z = -\tfrac{3}{2}(U) + \tfrac{1}{2}(\Gamma_\mu\Gamma_\nu U)\Gamma_\mu\Gamma_\nu + (\Gamma_\mu U)\Gamma_\mu$$
$$= -\tfrac{3}{2}(\Gamma_b U)\Gamma_b + \tfrac{1}{2}(\Gamma_\mu\Gamma_\nu\Gamma_b U)\Gamma_\mu\Gamma_\nu\Gamma_b + (\Gamma_\mu\Gamma_b U)\Gamma_\mu\Gamma_b \quad (2.17)$$

The symbol 'Z' in (2.17) is introduced for convenience.

From (2.3) and (2.4) we can write

$$U = U_n = \Gamma_\lambda A_{1_\lambda} A_2 A_3 \ldots A_n = \Gamma_\lambda A_{1_\lambda} U_{n-1} \quad (2.18)$$

Using relation (2.18) and the fact that the suffix 'b' occurring in (2.17) is arbitrary in its value and can always be adjusted to coincide with a particular value of the dummy suffix λ we obtain

$$Z = -\tfrac{3}{2}(U_{n-1})A_1 + \tfrac{1}{2}(\Gamma_\mu\Gamma_\nu U_{n-1})\Gamma_\mu\Gamma_\nu A_1 + (\Gamma_\mu U_{n-1})\Gamma_\mu A_1 \quad (2.19)$$

Repeatedly applying the procedure adopted in the derivation of (2.19) we can transfer the elements of U or U_n from the inside of the trace bracket to the outside in (2.17). We finally get

$$Z = 4U_R \quad (2.20)$$

where the reversed string U_R and also u_R are given by

$$U_R = A_n A_{n-1} \ldots A_1 \quad (2.21)$$

and

$$u_R = a_n a_{n-1} \ldots a_1 \quad (2.22)$$

From (2.17), (2.20) and the relation $(U_R) = (U)$ we obtain

$$4U = -\tfrac{3}{2}(U) + \tfrac{1}{2}\Gamma_\mu\Gamma_\nu(\Gamma_\nu\Gamma_\mu U) + (\Gamma_\mu U)\Gamma_\mu \quad (2.23)$$

Now relation (2.17) holds even without the trace brackets ( ). Then following the procedure adopted in the derivation of (2.20) we obtain

$$-\tfrac{3}{2}U + \tfrac{1}{2}\Gamma_\mu\Gamma_\nu U\Gamma_\mu\Gamma_\nu + \Gamma_\mu U\Gamma_\mu = -4U_R \quad (2.24)$$

Combining equations (2.17), (2.20) and (2.23) we get

$$\Gamma_\mu(\Gamma_\mu U) = 2U + 2U_R - (U) \quad (2.25)$$

Using relation (2.25) we obtain from (2.23) the following relation

$$\Gamma_\mu U\Gamma_\mu = 2(U) - U - 2U_R \quad (2.26)$$

With the help of relations (2.12), (2.15), (2.7), (2.8) and (2.26) we can show that for an even string u ($= u_n$, n is even) the scalar product $\gamma_i u\gamma_i$ is given by

$$\gamma_i u\gamma_i = \Gamma_\mu\bar{U}\Gamma_\mu - \Gamma_5\bar{U}\Gamma_5 \quad (2.27)$$
$$= 2(u) - u - \bar{u} - 2u_R \quad (2.28)$$

In a similar manner we obtain from (2.26) and (2.14) the scalar product involving an odd string, $\gamma_i u_{2m+1}\gamma_i = \gamma_i s\gamma_i$ as given below

$$\gamma_i s\gamma_i = 2(s\gamma_5)\gamma_5 - s + \bar{s} - 2s_R \tag{2.29}$$

Let us use the symbol s' to denote an odd strings for the special case when $a_{r_5} = 0$ for all values of r. Then we can obtain from (2.24) the following relation

$$\pm \tfrac{1}{2}\gamma_i\gamma_j s'\gamma_i\gamma_j + 2\gamma_i s'\gamma_i \mp 2s' = -4s'_R \tag{2.30}$$

We can establish from either (2.29) or (2.30) the following result

$$\gamma_i s'\gamma_i = -2s'_R \tag{2.31}$$

From (2.17), (2.20) and (2.23) we get

$$\begin{aligned}(\gamma_i\gamma_j u)(\gamma_i\gamma_j v) &= (\Gamma_i\Gamma_j U)(\Gamma_i\Gamma_j V)\\ &= 4(U)(V) + 2([U_R - U]\,[V + \bar{V}])\end{aligned} \tag{2.32}$$

where u, v, U and V are all even strings.

In relation (2.32) and in the following the elements of v and V are related in a manner similar to that which exists between u and U, as given by relations (2.13) and (2.15).

Using relations (2.25) and (2.7) we obtain the following result

$$\begin{aligned}(\gamma_i u)(\gamma_i v) &= (\Gamma_i U)(\Gamma_i V)\\ &= ([U + U_R]\,[V + \bar{V}])\end{aligned} \tag{2.33}$$

where u, v, U and V are all odd strings.

Let us multiply relation (2.25) from the left and also from the right by Γ_b and then add the two relations thus obtained. This enables us to obtain the following result

$$(U\Gamma_b)(V\Gamma_b) = ([U + U_R]\,[V + \Gamma_b V\Gamma_b]) - (U)(V) \tag{2.34}$$

where the suffix 'b' does not imply any summation and can assume any specific value from 1 to 5. Relation (2.34) helps us to determine $(u\gamma_b)(v\gamma_b)$. If we take u, v, U and V to be all odd strings then we have the result $(u\gamma_b)(v\gamma_{b'}) = (U\Gamma_b)(V\Gamma_b)$ where $b = 1, 2, 3$ or 4.

Setting $b = 5$ in (2.34) we obtain

$$\begin{aligned}(\gamma_5 u)(\gamma_5 v) = (\Gamma_5 U)(\Gamma_5 V) &= ([U + U_R]\,[V + \bar{V}]) - (U)(V)\\ &= (v\,[u + \bar{u}_R + \bar{u} + u_R]) - (u)(v)\end{aligned} \tag{2.35}$$

where u, v, U and V are all even strings.

We may note that the right-hand side of (2.35) does not involve any γ_5 matrix.

The relation (2.25) helps us to obtain a reduction formula for the evaluation of the trace (U_n) or (U) where $U_n = A_1 A_2 \ldots A_n$. Using (2.25) for the string $U_4 = A_1 A_2 A_3 A_4$ and writing $U_{n-4} = A_5 A_6 \ldots A_n$ we can obtain for odd n

$$(U_n) = (U_4 U_{n-4}) = \tfrac{1}{2}([U_4 - U_{4_R} + (U_4)]\, U_{n-4}) - \tfrac{1}{4}(U_4)(U_{n-4})$$
$$-\tfrac{1}{4} \sum_{r \geqslant 5}^{n} (-1)^r (U_4 A_r)(A_5 A_6 \ldots A_{r-1} A_{r+1} \ldots A_n) \quad (2.36)$$

where

$$\tfrac{1}{2}[U_4 - U_{4_R} + (U_4)] = A_1 . A_2 A_3 A_4 - A_1 . A_3 A_2 A_4 + A_1 . A_4 A_2 A_3$$
$$+ A_2 . A_3 A_1 A_4 - A_2 . A_4 A_1 A_3 + A_3 . A_4 A_1 A_2 \quad (2.37)$$

Next we derive in a similar manner another formula for (U_n) which is valid irrespective of whether n is odd or even. Putting $U_5 = A_1 A_2 A_3 A_4 A_5$, $U_{n-5} = A_6 A_7 \ldots A_n$ and using (2.25) for the string U_5 we get

$$(U_n) = (U_5 U_{n-5}) = \tfrac{1}{2}([U_4 - U_{4_R} + (U_4)]\, U_{n-4}) - \tfrac{1}{4}(U_4)(U_{n-4})$$
$$+ \tfrac{1}{4}(U_5)(U_{n-5}) + ([-A_5 . A_1 \{A_2 A_3 A_4\}$$
$$+ A_5 . A_2 \{A_1 A_3 A_4\} - A_5 . A_3 \{A_1 A_2 A_4\}$$
$$+ A_5 . A_4 \{A_1 A_2 A_3\}]\, U_{n-5}) \quad (2.38)$$

where terms of the type $\{A_2 A_3 A_4\}$ stand for

$$\{A_2 A_3 A_4\} = A_2 A_3 A_4 - A_2 . A_3 A_4 + A_2 . A_4 A_3 + A_3 . A_4 A_2$$

Relations similar to (2.36) and (2.38) have previously been derived by the author in a different manner. The relations (2.36) and (2.38) combined with (2.13) and (2.14) provide us with formulae for the determination of the traces (u_n) and $i(\gamma_5 u_n)$.

Several formulae for the evaluation of (u_n) and $(\gamma_5 u_n)$ when $a_{r_5} = 0$ for all values of the suffix r can be obtained from equations (2.36) and (2.38).

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